

# LOGARITHMIC KNOT INVARIANTS ARISING FROM RESTRICTED QUANTUM GROUPS

JUN MURAKAMI AND KIYOKAZU NAGATOMO

**ABSTRACT.** We construct knot invariants from the radical part of projective modules of the restricted quantum group  $\overline{\mathcal{U}}_q(sl_2)$  at  $q = \exp(\pi\sqrt{-1}/p)$ , and we also show a relation between these invariants and the colored Alexander invariants. These projective modules are related to logarithmic conformal field theories.

## 1. INTRODUCTION

Various knot invariants are constructed from the quantum  $R$ -matrix of the quantum group. However, most of them are constructed from semisimple algebras. Our concern in this note is constructing knot invariant arising from *non-semisimple* representations. We focus on the restricted quantum group  $\overline{\mathcal{U}}_q(sl_2)$  and construct knot invariant which is understood as a derivative of the colored Alexander invariant [1], [9].

Let  $Z$  and  $J$  be the center and the Jacobson radical of  $\overline{\mathcal{U}}_q(sl_2)$  respectively. Then  $Z$  is a direct sum of  $Z^{(s)}$  and  $Z^{(r)}$ , where  $Z^{(s)}$  is the subalgebra of  $Z$  generated by the primitive idempotents and  $Z^{(r)} = Z \cap J$ . Let  $K$  be a knot in  $S^3$ . By using the idea of the universal invariant in [10], we can associate an element  $z_K \in Z$  with  $K$ . Then  $z_K$  is expressed as

$$z_K = z_K^{(s)} + z_K^{(r)} \quad (z_K^{(s)} \in Z^{(s)}, z_K^{(r)} \in Z^{(r)}).$$

The first term  $z_K^{(s)}$  is a linear combination of the primitive idempotents and the coefficient of each idempotent corresponds to the colored Jones invariant. In this paper, we study about the knot invariants coming from  $z_K^{(r)}$ . The space  $Z^{(r)} = Z \cap J$  has a natural basis corresponding to the indecomposable modules, and the coefficients of  $z_K^{(r)}$  with respect to this basis are also knot invariants.

In the construction of  $z_K$ , we assume that  $K$  is a single component knot. For a multi-component link, the construction in this paper does not work well and we need another idea to extend  $z_K^{(r)}$  for link case.

A three-manifold invariant is constructed from the ‘integral’ of  $\overline{\mathcal{U}}_q(sl_2)$  [8], [4], [7], [11], which is defined for a finite dimensional Hopf algebra. Meanwhile, an action of  $SL(2, \mathbb{Z})$  on the center  $Z$  of  $\overline{\mathcal{U}}_q(sl_2)$  is given in [6] and [3]. By using our invariants constructed here, these two theories can be combined as the usual topological quantum field theory, e.g. [12], [2], related to  $\mathcal{U}_q(sl_2)$ . The detail will be given elsewhere.

We review the definition of  $\overline{\mathcal{U}}_q(sl_2)$  and its representations in Section 2. The construction of  $z_K^{(r)}$  is given in Section 3. In Sections 4 and 5, we show some property of the invariants coming from  $z_K$ , especially the relation to the colored Alexander invariant in [9].

## 2. RESTRICTED QUANTUM GROUPS

**2.1. Definition.** Let  $p \geq 2$  be a positive integer and  $q = \exp(\pi\sqrt{-1}/p)$ . The semi-restricted quantum group  $\widehat{\mathcal{U}}_q(sl_2)$  is the quotient of the usual quantum group  $\mathcal{U}_q(sl_2)$

defined by the following generators and relations as an algebra.

$$\begin{aligned}\widehat{\mathcal{U}}_q(sl_2) = \langle & K, K^{-1}, E, F \mid K K^{-1} = K^{-1} K = 1, \\ & K E K^{-1} = q^2 E, \quad K F K^{-1} = q^{-2} F, \quad E F - F E = \frac{K - K^{-1}}{q - q^{-1}}, \\ & E^p = F^p = 0 \rangle.\end{aligned}$$

The restricted quantum group  $\overline{\mathcal{U}}_q(sl_2)$  is obtained from  $\widehat{\mathcal{U}}_q(sl_2)$  by inquiring one more relation  $K^{2p} = 1$ . The coproduct, counit and antipode of  $\widehat{\mathcal{U}}_q(sl_2)$  and  $\overline{\mathcal{U}}_q(sl_2)$  are defined as follows.

$$\begin{aligned}\Delta(K) &= K \otimes K, \quad \Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \\ \epsilon(E) &= \epsilon(F) = 0, \quad \epsilon(K) = 1, \\ S(E) &= -E K^{-1}, \quad S(F) = -K F, \quad S(K) = K^{-1}.\end{aligned}$$

**2.2.  $R$ -matrix.** By introducing a symbol  $k$  such that  $k^2 = K$ , We can define an  $R$ -matrix  $R$  of  $\overline{\mathcal{U}}_q(sl_2)$  satisfying

$$R \Delta(X) R^{-1} = \overline{\Delta}(X),$$

where  $\overline{\Delta}(X) = \sum_i z_i \otimes y_i$  if  $\Delta(X) = \sum_i y_i \otimes z_i$ . The explicit form of  $R$  is given as follows.

$$(2.1) \quad R = q^{\frac{1}{2}H \times H} \sum_{n=0}^{p-1} \frac{(q - q^{-1})^n}{[n]!} q^{\frac{n(n-1)}{2}} (E^n \otimes F^n),$$

where  $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$  and  $[n]! = [n][n-1] \cdots [1]$ , and  $H$  is an element such that  $q^H = K$ .

**2.3. Irreducible modules.** The irreducible modules  $\mathcal{X}^\alpha(s)$  of  $\overline{\mathcal{U}}_q(sl_2)$  are labeled by  $\alpha = \pm 1$  and  $1 \leq s \leq p$ , and is spanned by weight vectors  $|s, n\rangle^\pm$ ,  $0 \leq n \leq s-1$  with the action of  $\overline{\mathcal{U}}_q(sl_2)$  given by

$$\begin{aligned}K |s, n\rangle^\pm &= \pm q^{s-1-2n} |s, n\rangle^\pm, \\ E |s, n\rangle^\pm &= \pm [n][s-n] |s, n-1\rangle^\pm, \\ F |s, n\rangle^\pm &= |s, n+1\rangle^\pm,\end{aligned}$$

where  $|s, s\rangle^\pm = 0$ .

**2.4. Projective modules.** Projective modules  $\mathcal{P}^\pm(s)$  of  $\overline{\mathcal{U}}_q(sl_2)$  which are fundamental to investigate the structure of the center of  $\overline{\mathcal{U}}_q(sl_2)$  are labeled by  $1 \leq s \leq p-1$ . Note that a  $\overline{\mathcal{U}}_q(sl_2)$  module is also naturally a  $\widehat{\mathcal{U}}_q(sl_2)$  module.

Let  $s$  be any integer  $1 \leq s \leq p-1$ . The projective module  $\mathcal{P}^+(s)$  has the basis

$$\{x_k^{(+,s)}, y_k^{(+,s)}\}_{0 \leq k \leq p-s-1} \cup \{a_n^{(+,s)}, b_n^{(+,s)}\}_{0 \leq n \leq s-1},$$

and the action of  $\overline{\mathcal{U}}_q(sl_2)$  is given by

$$\begin{aligned}K x_k^{(+,s)} &= -q^{p-s-1-2k} x_k^{(+,s)}, \quad K y_k^{(+,s)} = -q^{p-s-1-2k} y_k^{(+,s)}, \quad 0 \leq k \leq p-s-1, \\ K a_n^{(+,s)} &= q^{s-1-2n} a_n^{(+,s)}, \quad K b_n^{(+,s)} = q^{s-1-2n} b_n^{(+,s)}, \quad 0 \leq n \leq s-1, \\ E x_k^{(+,s)} &= -[k][p-s-k] x_{k-1}^{(+,s)}, \quad 0 \leq k \leq p-s-1, \quad (\text{with } x_{-1}^{(+,s)} = 0),\end{aligned}$$

$$\begin{aligned}
 E y_k^{(+,s)} &= \begin{cases} -[k][p-s-k] y_{k-1}^{(+,s)}, & 1 \leq k \leq p-s-1, \\ a_{s-1}^{(+,s)}, & k=0, \end{cases} \\
 E a_n^{(+,s)} &= [n][s-n] a_{n-1}^{(+,s)}, \quad 0 \leq n \leq s-1, \quad (\text{with } a_{-1}^{(+,s)} = 0), \\
 E b_n^{(+,s)} &= \begin{cases} [n][s-n] b_{n-1}^{(+,s)} + a_{n-1}^{(+,s)}, & 1 \leq n \leq s-1, \\ x_{p-s-1}^{(+,s)}, & n=0, \end{cases}
 \end{aligned}$$

and

$$\begin{aligned}
 F x_k^{(+,s)} &= \begin{cases} x_{k+1}^{(+,s)}, & 0 \leq k \leq p-s-2, \\ a_0^{(+,s)}, & k=p-s-1, \end{cases} \\
 F y_k^{(+,s)} &= y_{k+1}^{(+,s)}, \quad 0 \leq k \leq p-s-1, \quad (\text{with } y_{p-s}^{(+,s)} = 0), \\
 F a_n^{(+,s)} &= a_{n+1}^{(+,s)}, \quad 0 \leq n \leq s-1, \quad (\text{with } a_s^{(+,s)} = 0), \\
 F b_n^{(+,s)} &= \begin{cases} b_{n+1}^{(+,s)}, & 0 \leq n \leq s-2, \\ y_0^{(+,s)}, & n=s-1. \end{cases}
 \end{aligned}$$

Let  $s$  be an integer  $1 \leq s \leq p-1$ . The projective module  $\mathcal{P}^-(p-s)$  has the basis

$$\{x_k^{(-,s)}, y_k^{(-,s)}\}_{0 \leq k \leq p-s-1} \cup \{a_n^{(-,s)}, b_n^{(-,s)}\}_{0 \leq n \leq s-1},$$

and the action of  $\overline{\mathcal{U}}_q(sl_2)$  is given by

$$\begin{aligned}
 K x_k^{(-,s)} &= -q^{p-s-1-2k} x_k^{(+,s)}, \quad K y_k^{(+,s)} = -q^{p-s-1-2k} y_k^{(-,s)}, \quad 0 \leq k \leq p-s-1, \\
 K a_n^{(-,s)} &= q^{s-1-2n} a_n^{(-,s)}, \quad K b_n^{(-,s)} = q^{s-1-2n} b_n^{(-,s)}, \quad 0 \leq n \leq s-1, \\
 E x_k^{(-,s)} &= -[k][p-s-k] x_{k-1}^{(-,s)}, \quad 0 \leq k \leq p-s-1, \quad (\text{with } x_{-1}^{(-,s)} = 0)
 \end{aligned}$$

$$\begin{aligned}
 E y_k^{(-,s)} &= \begin{cases} -[k][p-s-k] y_{k-1}^{(-,s)} + x_{k-1}^{(-,s)}, & 1 \leq k \leq p-s-1, \\ a_{s-1}^{(-,s)}, & k=0, \end{cases} \\
 E a_n^{(-,s)} &= [n][s-n] a_{n-1}^{(-,s)}, \quad 0 \leq n \leq s-1, \quad (\text{with } a_{-1}^{(-,s)} = 0), \\
 E b_n^{(-,s)} &= \begin{cases} [n][s-n] b_{n-1}^{(-,s)}, & 1 \leq n \leq s-1, \\ x_{p-s-1}^{(-,s)}, & n=0, \end{cases}
 \end{aligned}$$

and

$$\begin{aligned}
 F x_k^{(-,s)} &= x_{k+1}^{(-,s)}, \quad 0 \leq k \leq p-s-1, \quad (\text{with } x_{p-s}^{(-,s)} = 0), \\
 F y_k^{(-,s)} &= \begin{cases} y_{k+1}^{(-,s)}, & 0 \leq k \leq p-s-2, \\ b_0^{(-,s)}, & k=p-s-1, \end{cases} \\
 F a_n^{(-,s)} &= \begin{cases} a_{n+1}^{(-,s)}, & 0 \leq n \leq s-2, \\ x_0^{(-,s)}, & n=s-1, \end{cases} \\
 F b_n^{(-,s)} &= b_{n+1}^{(-,s)}, \quad 0 \leq n \leq s-1, \quad (\text{with } b_s^{(-,s)} = 0).
 \end{aligned}$$

Note that the diagonal part is a direct sum of irreducible modules.

**2.5. Center.** The dimension of the center  $\mathcal{Z}$  of  $\overline{\mathcal{U}}_q(sl_2)$  is  $3p-1$ . The basis of  $\mathcal{Z}$  is given by the canonical central elements in [3] as follows: Two special primitive idempotents  $e_0$  and  $e_p$ , other primitive idempotents  $e_s$ , ( $1 \leq s \leq p-1$ ), and  $2(p-1)$  elements  $w_s^\pm$

$(1 \leq s \leq p-1)$  corresponding to the radical part. These basis satisfy the following relations.

$$(2.2) \quad \begin{aligned} e_s e_{s'} &= \delta_{s,s'} e_s, & s, s' &= 0, \dots, p, \\ e_s w_{s'}^\pm &= \delta_{s,s'} w_s^\pm, & 0 \leq s \leq p, 1 \leq s' \leq p-1, \\ w_s^\pm w_{s'}^\pm &= w_s^\pm w_{s'}^\mp = 0, & 1 \leq s, s' \leq p-1. \end{aligned}$$

### 3. LOGARITHMIC INVARIANTS OF KNOTS

**3.1. Knots and (1,1)-tangles.** In this paper, knots and tangles are oriented and framed. For a connected (1,1)-tangle  $T$ , let  $K_T$  be the knot obtained by joining the two open ends as in Figure 1. For two tangles  $T$  and  $T'$ , it is known that  $K_T$  and  $K_{T'}$  are isotopic as framed knots if and only if  $T$  and  $T'$  are isotopic as framed tangles. So, in the following, we sometimes mix up invariants of connected (1,1)-tangles and invariants of knots.

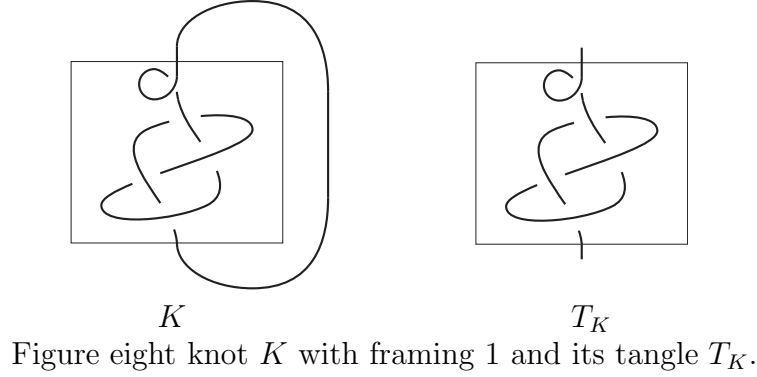


FIGURE 1. Cosure of a framed tangle.

**3.2. Framed braid.** Framed braid group on  $n$  strings  $FB_n$  is defined by the following generators and relations.

$$\begin{aligned} FB_n = \langle & \sigma_1, \sigma_2, \dots, \sigma_{n-1}, \tau_1, \tau_2, \dots, \tau_n \mid \\ & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad (i = 1, 2, \dots, n-2), \\ & \sigma_i \sigma_j = \sigma_j \sigma_i, \quad (|i-j| > 1), \\ & \tau_i^{\pm 1} \sigma_i = \sigma_i \tau_{i+1}^{\pm 1}, \quad \tau_{i+1}^{\pm 1} \sigma_i = \sigma_i \tau_i^{\pm 1}, \quad (i = 1, 2, \dots, n-1), \\ & \sigma_i \tau_j = \tau_j \sigma_i, \quad \tau_i \tau_j = \tau_j \tau_i \quad (|i-j| > 1) \rangle. \end{aligned}$$

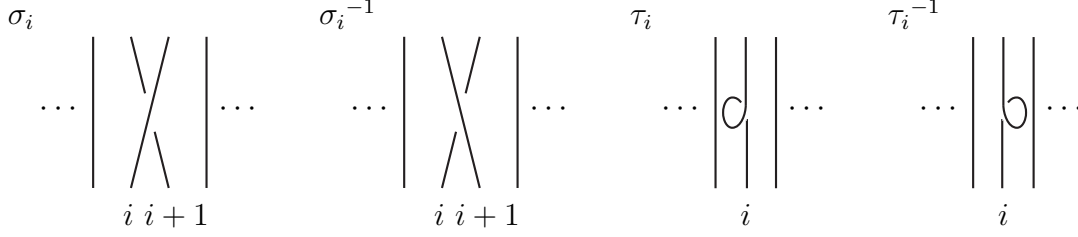
The generators  $\sigma_i^{\pm 1}$  correspond to the positive and negative crossings, and  $\tau_i^{\pm 1}$  represent the blackboard framing corresponding to the twist as in Figure 2. Let  $S_n$  be the symmetric group of  $n$  letters  $\{1, 2, \dots, n\}$ , and  $\pi$  be the group homomorphism from  $FB_n$  to  $S_n$  sending  $\sigma_i$  to the transposition  $(i, i+1)$  and  $\tau_i$  to the identity for  $i = 1, 2, \dots, n-1$ .

**3.3. Alexander's and Markov's theorems.** Then we have the framed versions of Alexander's and Markov's theorem as follows.

**Theorem 3.3.1.** *Any framed link is isotopic to the closure  $\hat{b}$  of some framed braid  $b \in FB_n$ .*

**Theorem 3.3.2.** *Two framed braids  $b_1, b_2$  have isotopic closures if and only if  $b_1$  can be transformed to  $b_2$  by a finite sequence of moves of the following two types.*

$$(i) \quad \beta_1 \beta_2 \longleftrightarrow \beta_2 \beta_1 \quad \text{for } \beta_1, \beta_2 \in FB_n.$$


 FIGURE 2. Generators of the framed braid group  $FB_n$ .

$$(ii) \quad \beta \tau_n^{\pm 1} \longleftrightarrow i(\beta) \sigma_n^{\pm 1} \quad \text{for } \beta \in FB_n \xrightarrow{i} FB_{n+1}.$$

**3.4. Representation of  $FB_n$  on  $\otimes^n \overline{\mathcal{U}}_q(sl_2)$ .** The universal  $R$ -matrix satisfies the Yang-Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} : \otimes^3 \overline{\mathcal{U}}_q(sl_2) \longrightarrow \otimes^3 \overline{\mathcal{U}}_q(sl_2),$$

where  $\otimes^3 \overline{\mathcal{U}}_q(sl_2) = \overline{\mathcal{U}}_q(sl_2) \otimes \overline{\mathcal{U}}_q(sl_2) \otimes \overline{\mathcal{U}}_q(sl_2)$  and  $R_{ij}$  acts on the  $i$ -th and  $j$ -th components of the tensor product. Let

$$R = \sum_i r'_i \otimes r''_i : \overline{\mathcal{U}}_q(sl_2) \otimes \overline{\mathcal{U}}_q(sl_2) \longrightarrow \overline{\mathcal{U}}_q(sl_2) \otimes \overline{\mathcal{U}}_q(sl_2),$$

and

$$v = \sum_i r''_i K^{N-1} r'_i : \overline{\mathcal{U}}_q(sl_2) \longrightarrow \overline{\mathcal{U}}_q(sl_2).$$

Then  $(\overline{\mathcal{U}}_q(sl_2), R, v)$  is a ribbon Hopf algebra with the ribbon element  $v$ . Therefore, we can define a homomorphism  $\rho$  from  $FB_n$  to  $\text{End}(\otimes^n \overline{\mathcal{U}}_q(sl_2))$  by

$$\begin{aligned} \rho(\sigma_i)(x_1 \otimes \dots \otimes x_i \otimes x_{i+1} \otimes \dots \otimes x_n) &= \sum_i x_1 \otimes \dots \otimes r'_i x_{i+1} \otimes r''_i x_i \otimes \dots \otimes x_n, \\ \rho(\tau_i)(x_1 \otimes \dots \otimes x_i \otimes \dots \otimes x_n) &= x_1 \otimes \dots \otimes v x_i \otimes \dots \otimes x_n. \end{aligned}$$

**3.5. Universal invariant.** Let  $K$  be a knot and let  $b_K$  be a framed braid whose closure is equivalent to  $K$ . Then  $\rho(b_K) \in \text{End}(\otimes^n \overline{\mathcal{U}}_q(sl_2))$  is expressed as follows.

$$\rho(b_K) = \sum_i a_{1,i} \otimes a_{2,i} \otimes \dots \otimes a_{n,i}.$$

Let  $T$  be a  $(1,1)$ -tangle corresponding to  $K$ . The element  $z_T \in \overline{\mathcal{U}}_q(sl_2)$  corresponding to  $T$  is defined by

$$z_T = a_{\pi(b_K)^{n-1}(1),i} K^{N-1} a_{\pi(b_K)^{n-2}(1),i} \dots a_{\pi(b_K)^2(1),i} K^{N-1} a_{\pi(b_K)(1),i} K^{N-1} a_{1,i}.$$

The element  $z_T \in \overline{\mathcal{U}}_q(sl_2)$  commutes with any elements of  $\overline{\mathcal{U}}_q(sl_2)$  and is in the center  $\mathcal{Z}$ . Therefore, we have

$$(3.1) \quad z_T = \sum_{s=0}^p (a_s(T) e_s + b_s^+(T) w_s^+ + b_s^-(T) w_s^-),$$

where  $a_s(T)$ ,  $b_s^\pm(T)$  are scalars and are invariants of the closure  $K$  of  $T$ . Hence we can also denote them by  $a_s(K)$ ,  $b_s^\pm(K)$ .

4. PROPERTIES OF  $a_s(K)$  AND  $b_s^\pm(K)$ 

We show some property of  $a_s(K)$  and  $b_s^\pm(K)$ .

**Theorem 4.0.1.** (1) For the connected sum of two knots  $K_1$  and  $K_2$ ,

$$a_s(K_1 \# K_2) = a_s(K_1) a_s(K_2), \quad b_s^\pm(K_1 \# K_2) = a_s(K_1) b_s^\pm(K_2) + b_s^\pm(K_1) a_s(K_2).$$

(2) For a knot  $K$ , the invariant  $a_s(K)$  ( $1 \leq s \leq p$ ) is equal to the colored Jones invariant  $J_s(K)$  corresponding to the  $s$ -dimensional irreducible module  $\mathcal{X}^+(s)$  normalized as

$$J_s(\text{unknot}) = 1.$$

*Proof.* For two tangles  $T_1, T_2$ , let  $T_1 \cdot T_2$  be the tangle obtained by joining  $T_2$  below  $T_1$ . Then, for two knots  $K_1$  and  $K_2$ ,  $T_{K_1} \cdot T_{K_2}$  is a tangle representing the connected sum  $K_1 \# K_2$ . Therefore, by using (2.2) and (3.1), we have

$$\begin{aligned} z_{K_1 \# K_2} &= z_{K_1} z_{K_2} \\ &= \sum_{s=0}^p (a_s(K_1) a_s(K_2) e_s + \\ &\quad (b_s^+(K_1) a_s(K_2) + a_s(K_1) b_s^+(K_2)) w_s^+ + (b_s^-(K_1) a_s(K_2) + a_s(K_1) b_s^-(K_2)) w_s^-) \end{aligned}$$

and we obtain (1).

The center  $e_s$  acts on  $\mathcal{X}(s)$  as identity, and the other basis  $e_t$  ( $t \neq s$ ), and  $w_t^\pm$  acts on  $\mathcal{X}(s)$  as zero. Hence  $a_s(K)$  corresponds to the scalar representing the action of  $z_K$  on  $\mathcal{X}(s)$ , which is equal to the colored Jones invariant and we get (2).  $\square$

## 5. RELATION TO THE COLORED ALEXANDER INVARIANT

**5.1. Relation.** Let  $K$  be a framed knot and  $T_K$  be the corresponding framed tangle. Let  $O_\lambda^p(T_K)$  be the scalar multiple of the colored Alexander invariant defined in [9]. Then we have the following.

**Theorem 5.1.1.** The invariants  $a_s(K)$ ,  $b_s^+(K)$ ,  $b_s^-(K)$  are given by the colored Alexander invariants as follows.

$$\begin{aligned} a_s(K) &= O_{s-1}^p(T_K), \quad 0 \leq s \leq p, \\ b_s^+(K) &= -\frac{p \sin^2 \frac{\pi}{p}}{\pi \sin \frac{\pi s}{p}} \left( \left. \frac{d O_\lambda^p(T_K)}{d\lambda} \right|_{\lambda=2p-s-1} - \left. \frac{d O_\lambda^p(T_K)}{d\lambda} \right|_{\lambda=s-1} \right), \quad 1 \leq s \leq p-1, \\ b_s^-(K) &= \frac{p \sin^2 \frac{\pi}{p}}{\pi \sin \frac{\pi s}{p}} \left( \left. \frac{d O_\lambda^p(T_K)}{d\lambda} \right|_{\lambda=s-1} - \left. \frac{d O_\lambda^p(T_K)}{d\lambda} \right|_{\lambda=-s-1} \right), \quad 1 \leq s \leq p-1. \end{aligned}$$

Before proving the above, we introduce some representations of  $\widehat{\mathcal{U}}_q(sl_2)$ .

**5.2. Non-integral representations.** We introduce highest weight representations of  $\widehat{\mathcal{U}}_q(sl_2)$  for non-integral weights and obtain the projective modules  $\mathcal{P}^\pm$  as a specialization of certain non-irreducible module.

First, we define the irreducible module for non-integer number  $\lambda$  as follows. Let  $\mathcal{X}(\lambda)$  be the  $\widehat{\mathcal{U}}_q(sl_2)$  module spanned by weight vectors  $v_n^\lambda$ ,  $0 \leq n \leq p-1$ . The action of  $\widehat{\mathcal{U}}_q(sl_2)$  to  $\mathcal{X}(\lambda)$  is given by

$$K v_n^\lambda = q^{\lambda-1-2n} v_n^\lambda, \quad E v_n^\lambda = [n][\lambda-n] v_{n-1}^\lambda, \quad F v_n^\lambda = v_{n+1}^\lambda,$$

where  $v_p^\lambda = 0$ .

Next, we define a non-irreducible module which is isomorphic to direct sum of two non-integral highest modules. Let  $t$  be an integer with  $1 \leq s \leq p$  and  $\mathcal{V}(\lambda, s)$  be the  $\widehat{\mathcal{U}}_q(sl_2)$  module which is spanned by weight vectors  $c_j^{(\lambda, s)}$  and  $d_j^{(\lambda, s)}$  for  $0 \leq j \leq p-1$ . The action of  $\widehat{\mathcal{U}}_q(sl_2)$  is given by

$$\begin{aligned} K c_n^{(\lambda, s)} &= q^{\lambda-1-2n} c_n^{(\lambda, s)}, & K d_n^{(\lambda, s)} &= q^{\lambda-1-2s-2n} d_n^{(\lambda, s)}, & 0 \leq n \leq p-1, \\ E c_n^{(\lambda, s)} &= \begin{cases} 0, & n = 0, \\ [n][\lambda - n] c_{n-1}^{(\lambda, s)}, & 1 \leq n \leq p-1, \end{cases} \\ E d_n^{(\lambda, s)} &= \begin{cases} c_{s-1}^{(\lambda, s)}, & n = 0, \\ [n][\lambda - 2s - n] d_{n-1}^{(\lambda, s)} + c_{n+s-1}^{(\lambda, s)}, & 1 \leq n \leq p-s, \\ [n][\lambda - 2s - n] d_{n-1}^{(\lambda, s)}, & p-s+1 \leq n \leq p-1, \end{cases} \\ F c_n^{(\lambda, t)} &= \begin{cases} c_{n+1}^{(\lambda, s)}, & 0 \leq n \leq p-2, \\ 0, & n = p-1, \end{cases} \\ F d_n^{(\lambda, s)} &= \begin{cases} d_{n+1}^{(\lambda, s)}, & 0 \leq n \leq p-2, \\ 0, & n = p-1. \end{cases} \end{aligned}$$

**5.3. Colored Alexander invariant.** Let  $K$  be a framed knot,  $T_K$  be the corresponding framed tangle, and  $\hat{z}_K$  be the corresponding central element in the semi-restricted quantum group  $\widehat{\mathcal{U}}_q(sl_2)$  which is defined as  $z_K$  by using the universal R-matrix of  $\widehat{\mathcal{U}}_q(sl_2)$  given by (2.1). Let  $Z_K^{(\lambda, s)}$  be the representation matrix of  $\hat{z}_K$  on  $\mathcal{V}(\lambda, s)$  with respect to the above basis  $\{c_n^{(\lambda, s)}, d_n^{(\lambda, s)}; 0 \leq n \leq p-1\}$ . Then the diagonal element corresponding to  $c_n^{(\lambda, s)}$  and  $d_n^{(\lambda, s)}$  ( $0 \leq n \leq p-1$ ) are equal to  $O_{\lambda-1}^p(T_K)$  and  $O_{\lambda-1-2s}^p(T_K)$  respectively, where  $O_{\lambda}^p(T_K)$  is given in [9] as the scalar corresponding to the tangle  $T_K$ . Note that  $O_{\lambda}^p(T_K)$  is a scalar multiple of the colored alexander invariant  $\Phi_K^p(\lambda)$  and  $O_{\lambda}^p(T_K)$  itself is also an invariant of  $K$  if  $K$  is a single component knot.

**5.4. Proof of Theorem 4.** The matrix  $Z_K^{(\lambda, s)}$  has off-diagonal elements at  $(c_{n+s}, d_n)$  components for  $0 \leq n \leq p-s$ . Let  $x$  be the  $(c_s, d_0)$  component of  $Z_K^{(\lambda, s)}$ . Then

$$(5.1) \quad \begin{aligned} Z_K^{(\lambda, s)} c_s^{(\lambda, s)} &= O_{\lambda-1}^p(T_K) c_s^{(\lambda, s)}, \\ Z_K^{(\lambda, s)} d_0^{(\lambda, s)} &= O_{\lambda-1-2s}^p(T_K) d_0^{(\lambda, s)} + x c_s^{(\lambda, s)}. \end{aligned}$$

Let

$$h = c_s^{(\lambda, s)} - [s][\lambda - s] d_0^{(\lambda, s)}.$$

Then  $Eh = 0$  and so  $h$  is a highest weight vector of weight  $\lambda - 1 - 2s$ . Therefore, on the one hand,

$$Z_K^{(\lambda, s)} h = O_{\lambda-1-2s}^p(T) h = O_{\lambda-1-2s}^p(T) c_s^{(\lambda, s)} - O_{\lambda-1-2s}^p(T) [s][\lambda - s] d_0^{(\lambda, s)}.$$

On the other hand, from (5.1),

$$Z_K^{(\lambda, s)} h = O_{\lambda-1}^p(T) c_0^{(\lambda, s)} - [s][\lambda - s] x c_0^{(\lambda, s)} - O_{\lambda-1-2s}^p(T) [s][\lambda - s] d_0^{(\lambda, s)}.$$

Thus we have

$$(5.2) \quad O_{\lambda-2s-1}^p(T) = O_{\lambda-1}^p(T) - [s][\lambda - s]x,$$

and then

$$x = \frac{O_{\lambda-1}^p(T) - O_{\lambda-1-2s}^p(T)}{[s][\lambda-s]}.$$

Hence, we get

$$(5.3) \quad \lim_{\lambda \rightarrow s+mp} x = (-1)^m \frac{p \sin \frac{\pi}{p}}{\pi [s]} \left( \frac{d O_{\lambda-1}^p(T)}{d\lambda} \Big|_{\lambda=s+mp} - \frac{d O_{\lambda-1-2s}^p(T)}{d\lambda} \Big|_{\lambda=s+mp} \right), \quad m \in \mathbf{Z}.$$

The projective module  $\mathcal{P}^+(s)$  is identical to  $\mathcal{Y}(2p-s, p-s)$  by the correspondence of the basis  $x_n^{(+,s)} \mapsto c_n^{(2p-s, p-s)}$ ,  $a_n^{(+,s)} \mapsto c_{n+p-s}^{(2p-s, p-s)}$ ,  $b_n^{(+,s)} \mapsto d_n^{(2p-s, p-s)}$ ,  $y_n^{(+,s)} \mapsto d_{n+s}^{(2p-s, p-s)}$ . Therefore, by substituting  $2p-s$  to  $\lambda$ ,  $p-s$  to  $s$ , and 1 to  $m$  for (5.3), we have

$$a_s(T) = O_{2p-s-1}^p(T),$$

and

$$b_s^+(T) = -\frac{p \sin^2 \frac{\pi}{p}}{\pi \sin \frac{\pi s}{p}} \left( \frac{d O_{\lambda}^p(T)}{d\lambda} \Big|_{\lambda=2p-s-1} - \frac{d O_{\lambda}^p(T)}{d\lambda} \Big|_{\lambda=s-1} \right), \quad 1 \leq s \leq p-1.$$

Similarly, the projective module  $\mathcal{P}^-(p-s)$  is identical to  $\mathcal{Y}(s, s)$  by the correspondence of the basis  $a_n^{(-,s)} \mapsto c_n^{(s, s)}$ ,  $x_n^{(-,s)} \mapsto c_{n+s}^{(s, s)}$ ,  $y_n^{(-,s)} \mapsto d_n^{(s, s)}$ ,  $b_n^{(-,s)} \mapsto d_{n+p-s}^{(s, s)}$ . Hence we have

$$b_{p-s}^-(T) = \frac{p \sin^2 \frac{\pi}{p}}{\pi \sin \frac{\pi s}{p}} \left( \frac{d O_{\lambda}^p(T)}{d\lambda} \Big|_{\lambda=s-1} - \frac{d O_{\lambda}^p(T)}{d\lambda} \Big|_{\lambda=-s-1} \right), \quad 1 \leq s \leq p-1.$$

The formula (5.2) implies that

$$O_{\lambda}^{mp+s-1}(T) = O_{\lambda}^{mp-s-1}(T).$$

By putting  $m = 1$  and  $s = p-s$ , we have

$$a_s(T) = O_{\lambda}^{2p-s-1}(T) = O_{\lambda}^{s-1}(T).$$

□

## REFERENCES

- [1] Y. Akutsu, T. Deguchi and T. Ohtsuki, Invariants of colored links, *J. Knot Theory Ramifications* **1** (1992) 161–184.
- [2] B. Bakalov and A. Kirillov Jr., *Lectures on tensor categories and modular functors*, University Lecture Series, **21** (American Mathematical Society, Providence, RI, 2001).
- [3] B. L. Feigin, A. M. Gainutdinov, A. M. Semikhatov and I. Yu. Tipunin, Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center, *Comm. Math. Phys.* **265** (2006) 47–93.
- [4] G. Hennings, Invariants of links and 3-manifolds obtained from Hopf algebras, *J. London Math. Soc.* (2) **54** (1996) 594–624.
- [5] M. Jimbo, T. Miwa and Y. Takeyama, Counting minimal form factors of the restricted sine-Gordon model, *Mosc. Math. J.* **4** (2004) 787–846, 981.
- [6] T. Kerler, Mapping class group actions on quantum doubles, *Comm. Math. Phys.* **168** (1995) 353–388.
- [7] L. Kauffman and D. Radford, Invariants of 3-manifolds derived from finite-dimensional Hopf algebras, *J. Knot Theory Ramifications* **4** (1995) 131–162.
- [8] G. Kuperberg, Involutory Hopf algebras and 3-manifold invariants, *Internat. J. Math.* **2** (1991) 41–66.
- [9] J. Murakami, Colored Alexander invariants and cone manifolds, to appear in *Osaka J. Math.*
- [10] T. Ohtsuki, Colored ribbon Hopf algebras and universal invariants of framed links, *J. Knot Theory Ramifications* **2** (1993) 211–232.

- [11] T. Ohtsuki, Invariants of 3-manifolds derived from universal invariants of framed links, *Math. Proc. Cambridge Philos. Soc.* **117** (1995) 259–273.
- [12] V. Turaev, *Quantum invariants of knots and 3-manifolds*, (de Gruyter, Berlin, 1994).

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ENGINEERING, WASEDA UNIVERSITY,  
3-4-1 OHKUBO, SHINJYUKU-KU, TOKYO 169-8555, JAPAN., E-MAIL: MURAKAMI@WASEDA.JP

DEPARTMENT OF PURE AND APPLIED MATHEMATICS, GRADUATE SCHOOL OF INFORMATION SCI-  
ENCE AND TECHNOLOGY, OSAKA UNIVERSITY, TOYONAKA, OSAKA 560-0043, JAPAN.